CERN-TH/2000-380

On the Embedding of Space-Time Symmetries into Simple Superalgebras

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Abstract

We explore the embedding of Spin groups of arbitrary dimension and signature into simple superalgebras in the case of extended supersymmetry. The R-symmetry, which generically is not compact, can be chosen compact for all the cases that are congruent mod 8 to the physical conformal algebra so($D-2$,2), $D \geq 3$. An so$(1,1)$ grading of the superalgebra is found in all cases. Central extensions of super translation algebras are studied in this framework.

1 Introduction

Supersymmetry algebras in higher dimensions [1, 2] have been the subject of investigation in recent years in connection with their relation to string and M theories [3]. The AdS-CFT correspondence (for a review, see Ref. [4]), which describes a duality between world-volume brane theories and AdS supergravities, has renewed the interest in superconformal field theories and their underlying superalgebras [5, 6, 7, 8]. It has also been conjectured [3, 5, 6, 7, 9, 10] that brane superalgebras are closely related to the extension of superconformal algebras to larger algebras including antisymmetric tensor generators.

The most remarkable example is $\mathrm{osp}(1|32,\mathbb{R})$, with 32 real spinor charges. By contraction [11] it gives the M-theory superalgebra (super Poincaré in an 11-dimensional space of signature $(10,1)$), with the maximal central extension (two and five brane charges). It also can be seen as the super conformal algebra in a space of dimension $D=10$ and signature $(9,1)$ [5, 6]. The conformal group in $D=10$ acts linearly on a space of dimension 12, $(10,2)$, and the spinor charge is a real Weyl spinor in this space. The (1,0) chiral $D=10$ super Poincaré algebra with a five-brane central charge is a subalgebra of $\mathrm{osp}(1|32,\mathbb{R})$.

Moving to one dimension more, the M-theory super Poincaré algebra (with two and five-brane charges) can be regarded as a subalgebra of the superconformal algebra of a space time of dimension 11. This superalgebra is $\mathrm{osp}(1|64,\mathbb{R})$, whose odd part (64 spinorial charges) forms a real spinor in a space of dimension 13, $(11,2)$ [12, 13]. This example is relevant because it has been shown that M-theory can be regarded as a phase of an $\mathrm{osp}(1|64,\mathbb{R})$ theory [14]. The full symmetry is spontaneously broken, so it is realized non linearly. Some attempts to derive M-theory from a gauge theory of the group $\mathrm{Sp}(32,\mathbb{R})$ have been made in Ref. [10]

In this paper we generalize the classification of space-time superalgebras given in Ref. [13], in terms of the dimension D and signature ρ of space time, to extended supersymmetry algebras with an R-symmetry group acting on the conformal spinors.

Other than brane superalgebras, the present results are in relation with different versions of supersymmetric theories with $D \leq 11$, in particular with the theories introduced by Hull [15], M, M* and M', which are formulated in 11 dimensional space-times with signatures $(s,t) = (10,1),(9,2),(6,5)$ respectively. Notice that the signature for all these theories is $\rho = s-t = \pm 1$

mod 8. Our analysis implies that the superalgebras appearing in them can be seen as a contraction of the very same $\mathrm{osp}(1|32,\mathbb{R})$ algebra or as a subalgebra of $\mathrm{osp}(1|64,\mathbb{R})$. This suggests that they are all different phases of the same theory [15, 16].

The present study evidences the deep relation between the R-symmetry algebra and the space-time algebra (the $\mathrm{Spin(s,t)}$-algebra which embeds $\mathrm{so(s,t)}$, according to the notation of [13]) that appear as simple factors in the bosonic part of the superconformal algebra. For the signature of the physical conformal group, $(s,t) = (D-2,2)$ a compact R-symmetry group is allowed. For the Euclidean superconformal algebra, that is, for signature $(s,t) = (D-1,1)$ only non compact R-symmetry groups are allowed. This implies that a proper treatment of Euclidean supersymmetric theories require superalgebras with non compact R-symmetry, as observed long ago by Zumino [17], and recently discussed in [18, 19, 20].

The paper is organized as follows. In Section 2 we extend the analysis of ref. [13] to N supersymmetries. In Section 3. we show how, in any dimension, the superconformal algebra has an so(1,1) grading. The super Poincaré algebra emerges as a non semisimple subalgebra of the superconformal algebra. In Section 4. we consider N-extended super Poincaré algebras with maximal central extension.

2 Conformal superalgebras with N supersymmetries

We recall here briefly the formalism used in Ref. [13]. Let S be a complex vector space. A conjugation of S is an antilinear map $\sigma : S \rightarrow S$,

$$\sigma(av) = a^*\psi(v), \qquad a \in \mathbb{C},\ v \in S,$$

such that $\sigma^2 = 1$. The set

$$S^\sigma = \{v \in S | \sigma(v) = v\}$$

is a real vector space. If instead $\sigma^2 = -1$ we say that σ is a pseudoconjugation of S, and the condition $\sigma(v) = v$ is inconsistent.

Suppose now that S is a $\mathcal{G}$-module, for a complex Lie algebra $\mathcal{G}$. Let $\sigma : S \rightarrow S$ be an antilinear map satisfying $\sigma^2 = \pm 1$ (that is, σ is a conjugation

or a pseudoconjugation on S). Then the map

$$\psi(X) = \sigma \circ X \circ \sigma^{-1}$$

is a conjugation of $\mathcal{G}$. If S is irreducible one can prove that $\mathcal{G}^\psi$ is a real form of $\mathcal{G}$. The action of $\mathcal{G}^\psi$ on S commutes with σ.

The real forms of classical Lie algebras arise in this way, except for the unitary algebras $\mathrm{su}(p,q)$. The algebras $\mathrm{sl}(n,\mathbb{R})$, $\mathrm{so}(p,q)$, $\mathrm{sp}(2p,\mathbb{R})$ correspond to conjugations on the space of the fundamental representation and the algebras $\mathrm{su}^*(2p)$, $\mathrm{so}^*(2p)$ and $\mathrm{usp}(2p,2q)$ correspond to pseudoconjugations.

Let V be a real vector space of dimension $D = s+t$ with a non degenerate symmetric bilinear form of signature $\rho = s-t$. We denote by $\mathrm{Spin}(s,t)$ the corresponding spin group. Spinor representations are linear, orthogonal or symplectic depending on the existence of a $\mathrm{Spin}(s,t)$-invariant bilinear form, which can be symmetric or antisymmetric. The existence and symmetry of the bilinear form depend exclusively on the dimension D modulo 8 [21]. The complexified group $\mathrm{Spin}(s,t)^{\mathbb{C}}$ is then embedded into a classical Lie group, $\mathrm{Sl}(n,\mathbb{C})$, $\mathrm{SO}(n,\mathbb{C})$ or $\mathrm{Sp}(n,\mathbb{C})$ where n is the dimension of the spinor representation. Denoting by $D = D_0 \bmod(8)$ and $\rho = \rho_0 \bmod(8)$, we have that for $D_0 = 2,6$ the spinors are linear, for $D_0 = 0,1,7$ they are orthogonal and for $D_0 = 3,4,5$ they are symplectic.

The existence of a conjugation or a pseudoconjugation in the space S of the spinor representation (commuting with the action of the orthogonal algebra) depends on the signature ρ mod 8. We have then that $\mathrm{o}(s,t)$ is embedded into a real form of the above algebra, which will be determined in terms of ρ. This real algebra is called the $\mathrm{Spin}(s,t)$ algebra. In Table 1 the real form for each case is listed [13]. The odd and even cases are treated separately.

We consider now superalgebras containing $\mathrm{so}(s,t)$ in the even part and whose odd part consists on one or more copies of the spinor representation of $\mathrm{so}(s,t)$. If the superalgebra is required to be simple, then the anticommutators of the odd charges generate the whole even subalgebra. In particular, the generators of the orthogonal group in the spinor representation must have the appropriate symmetry to appear in the right hand side of the anticommutator of two spinors. This analysis was carried out in Ref. [13] for $N=1$ supersymmetry. The smallest simple superalgebra containing $\mathrm{so}(s,t)$ has even part equal to the $\mathrm{Spin}(s,t)$ algebra, listed in Table 1, times an R-symmetry algebra which is $\mathrm{usp}(2)$ or $\mathrm{so}^*(2)$ (quaternionic case) or $\mathrm{u}(1)$

Orthogonal $D_0 = 1, 7$	Real, $\rho_0 = 1, 7$	$\mathrm{so}(2^{\frac{(D-1)}{2}}, \mathbb{R})$ if $D = \rho$
		$\mathrm{so}(2^{\frac{(D-1)}{2}-1}, 2^{\frac{(D-1)}{2}-1})$ if $D \neq \rho$
	Quaternionic, $\rho_0 = 3, 5$	$\mathrm{so}^*(2^{\frac{(D-1)}{2}})$
Symplectic $D_0 = 3, 5$	Real, $\rho_0 = 1, 7$	$\mathrm{sp}(2^{\frac{(D-1)}{2}}, \mathbb{R})$
	Quaternionic, $\rho_0 = 3, 5$	$\mathrm{usp}(2^{\frac{(D-1)}{2}}, \mathbb{R})$ if $D = \rho$
		$\mathrm{usp}(2^{\frac{(D-1)}{2}-1}, 2^{\frac{(D-1)}{2}-1})$ if $D \neq \rho$
Orthogonal $D_0 = 0$	Real, $\rho_0 = 0$	$\mathrm{so}(2^{\frac{D}{2}-1}, \mathbb{R})$ if $D = \rho$
		$\mathrm{so}(2^{\frac{D}{2}-2}, 2^{\frac{D}{2}-2})$ if $D \neq \rho$
	Quaternionic, $\rho_0 = 4$	$\mathrm{so}^*(2^{\frac{D}{2}-1})$
	Complex, $\rho_0 = 2, 6$	$\mathrm{so}(2^{\frac{D}{2}-1}, \mathbb{C})_{\mathbb{R}}$
Symplectic $D_0 = 4$	Real, $\rho_0 = 0$	$\mathrm{sp}(2^{\frac{D}{2}-1}, \mathbb{R})$
	Quaternionic, $\rho_0 = 4$	$\mathrm{usp}(2^{\frac{D}{2}-1}, \mathbb{R})$ if $D = \rho$
		$\mathrm{usp}(2^{\frac{D}{2}-2}, 2^{\frac{D}{2}-2})$ if $D \neq \rho$
	Complex, $\rho_0 = 2, 6$	$\mathrm{sp}(2^{\frac{D}{2}-1}, \mathbb{C})_{\mathbb{R}}$
Linear $D_0 = 2, 6$	Real, $\rho_0 = 0$	$\mathrm{sl}(2^{\frac{D}{2}-1}, \mathbb{R})$
	Quaternionic, $\rho_0 = 4$	$\mathrm{su}^*(2^{\frac{D}{2}-1})$
	Complex, $\rho_0 = 2, 6$	$\mathrm{su}(2^{\frac{D}{2}-1})$ if $D = \rho$
		$\mathrm{su}(2^{\frac{D}{2}-2}, 2^{\frac{D}{2}-2})$ if $D \neq \rho$

Table 1: $\mathrm{Spin}(s, t)$ algebras.

(complex case)[1]. This superalgebra is called $\mathrm{Spin}(s, t)$ superalgebra. The same structure appears in the case of extended supersymmetry, although the generalization is not completely straightforward since new cases appear due to the presence of the internal index space.

The odd part of the superalgebra is, as a vector space, a tensor product $S \otimes W$ of the spinor representation space S with the R-symmetry space W. If the spin algebra is a symplectic algebra, the R-symmetry is an orthogonal algebra and they build an orthosymplectic algebra. If the spin algebra is orthogonal, then the R-symmetry is symplectic and they give an orthosymplectic algebra with the roles of the orthogonal and symplectic groups in-

[1] Except for $D = 7$, $\rho = 3$, where a smaller simple superalgebra, the exceptional superalgebra $f(4)$, contains the orthogonal group in its even part.

terchanged. If the spin algebra is a linear (unitary) algebra, then the R-symmetry is also a linear (unitary) algebra and they build a superalgebra form the linear (unitary) series.

In order to obtain a real superalgebra, a conjugation commuting with the action of the even part of the superalgebra must exist in the total space $S \otimes W$. If the spinor is real then there is a conjugation σ_S of S commuting with the action of the $\mathrm{Spin}(s,t)$ algebra. The R-symmetry factor which acts on W also commutes with a conjugation, σ_W. If the spinor is quaternionic then there is a pseudoconjugation σ_S on S, and there is also a pseudoconjugation on W commuting with the R-symmetry. Then $\sigma_S \otimes \sigma_W$ is a conjugation in the total space. If the spinor is complex, the $\mathrm{Spin}(V)$ algebra is either $\mathrm{su}(p,q)$ or a complex group (symplectic or orthogonal). In the first case the R-symmetry is also $\mathrm{su}(m,n)$ and in the second case the R-symmetry is a complex group (orthogonal or symplectic respectively). The real representation is obtained by taking the complex vector space as a real one of twice the dimension.

As an example, we compute the case $D_0 = 1,7$, $\rho_0 = 1,7$. The spinors are orthogonal and real. The anticommutator of two odd generators is of the form

$$\{Q^i_\alpha, Q^j_\beta\} = \sum_k A^{ij} \gamma_{\alpha\beta}^{[\mu_1 \cdots \mu_k]} Z_{[\mu_1 \cdots \mu_k]}. \tag{1}$$

The generators of the orthogonal group are $Z_{[\mu_1 \mu_2]}$. For $N = 1$ the factor A^{ij} is not present. Then, in order to have $Z_{[\mu_1 \mu_2]}$ in the right hand side of (1), the morphism $\gamma_{\alpha\beta}^{[\mu_1 \mu_2]}$ must be symmetric. Since it is antisymmetric, there is no superconformal algebra in this case. For $N > 1$, one can choose an antisymmetric matrix $A^{ij} = \epsilon^{ij}$ and the orthogonal generators are allowed in the right hand side of (1). If follows that the R-symmetry group is $\mathrm{Sp}(2N, \mathbb{R})$.

In Table 2 we list the R-symmetry groups and the $\mathrm{Spin}(s,t)$ superalgebras. The compact cases $D = \rho$ are not listed but they are immediate. We mark with the symbol "$\circ$" the cases that do not arise in the non extended case.

The cases marked with a symbol "$\star$" allow the possibility of a compact R-symmetry group. They correspond to $\mathrm{so}(s,2)$, that is the physical conformal groups.

	D_0	ρ_0	R-symmetry	Spin(s,t) superalgebra
$\circ$	1,7	1,7	$\mathrm{sp}(2N,\mathbb{R})$	$\mathrm{osp}(2^{\frac{D-3}{2}},2^{\frac{D-3}{2}}\|2N,\mathbb{R})$
$\star$	1,7	3,5	$\mathrm{usp}(2N-2q,2q)$	$\mathrm{osp}(2^{\frac{D-1}{2}}{}^*\|2N-2q,2q)$
$\star$	3,5	1,7	$\mathrm{so}(N-q,q)$	$\mathrm{osp}(N-q,q\|2^{\frac{D-1}{2}})$
	3,5	3,5	$\mathrm{so}^*(2N)$	$\mathrm{osp}(2N^*\|2^{\frac{D-3}{2}},2^{\frac{D-3}{2}})$
$\circ$	0	0	$\mathrm{sp}(2N,\mathbb{R})$	$\mathrm{osp}(2^{\frac{D-4}{2}},2^{\frac{D-4}{2}}\|2N)$
$\circ$	0	2,6	$\mathrm{sp}(2N,\mathbb{C})_\mathbb{R}$	$\mathrm{osp}(2^{\frac{D-2}{2}}\|2N,\mathbb{C})_\mathbb{R}$
$\star$	0	4	$\mathrm{usp}(2N-2q,2q)$	$\mathrm{osp}(2^{\frac{D-2}{2}}{}^*\|2N-2q,2q)$
	2,6	0	$\mathrm{sl}(N,\mathbb{R})$	$\mathrm{sl}(2^{\frac{D-2}{2}}\|N,\mathbb{R})$
$\star$	2,6	2,6	$\mathrm{su}(N-q,q)$	$\mathrm{su}(2^{\frac{D-4}{2}},2^{\frac{D-4}{2}}\|N-q,q)$
$\circ$	2,6	4	$\mathrm{su}^*(2N,\mathbb{R})$	$\mathrm{su}(2^{\frac{D-2}{2}}\|2N)^*$
$\star$	4	0	$\mathrm{so}(N-q,q)$	$\mathrm{osp}(N-q,q\|2^{\frac{D-2}{2}})$
	4	2,6	$\mathrm{so}(N,\mathbb{C})_\mathbb{R}$	$\mathrm{osp}(N\|2^{\frac{D-2}{2}},\mathbb{C})_\mathbb{R}$
	4	4	$\mathrm{so}^*(2N)$	$\mathrm{osp}(2N^*\|2^{\frac{D-4}{2}},2^{\frac{D-4}{2}})$

Table 2: Spin(s,t) superalgebras.

3 so$(1,1)$ grading of the Spin(s,t) superalgebra

Let $\mathcal{G}^k$ be a compact semisimple Lie algebra, its complexification being $\mathcal{G}^c$. Let $\theta:\mathcal{G}^k\mapsto\mathcal{G}^k$ be an involutive automorphism, $\theta^2=1$. $\mathcal{G}^k$ splits into two eigenspaces,

$$\mathcal{G}^k=\mathcal{K}+\mathcal{P}.$$

$\mathcal{K}$ is the eigenspace with eigenvalue $+1$ of θ, and $\mathcal{P}$ the eigenspace with eigenvalue -1. The vector space

$$\mathcal{G}=\mathcal{L}_0+i\mathcal{P} \tag{2}$$

is a non compact real form of $\mathcal{G}^c$. (2) is called a *Cartan decomposition* of $\mathcal{G}$ and the procedure is known as the Weyl unitary trick. All the Cartan decompositions are listed in Ref.[22]. $\mathcal{L}_0$ is the maximal compactly embedded subalgebra of $\mathcal{G}$, and $\mathcal{P}$ carries an irreducible representation of $\mathcal{L}_0$.

If $\mathcal{G}$ is simple, then the algebra $\mathcal{K}$ is either semisimple or is a semisimple algebra plus a u(1) factor. For example, let $\mathcal{G}^k=\mathrm{so(p+q)}$ and $\theta_{p,q}(X)=$

$I_{p,q}XI_{p,q}$, where

$$I_{p,q} = \begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix}.$$

Then $\mathcal{K} = \mathrm{so}(p) \oplus \mathrm{so(q)}$. $\mathcal{P}$ is the bifundamental representation of $\mathrm{so}(p) \oplus \mathrm{so(q)}$, $(\mathbf{p}, \mathbf{q})$.

One can apply the Weyl unitary trick to $\mathcal{G}$ with respect to a Cartan involution θ' that commutes with θ. $\mathcal{G}$ splits into four eigenspaces of the simultaneous eigenvalues of θ and θ',

$$\mathcal{G} = \mathcal{G}^{++} + \mathcal{G}^{+-} + i\mathcal{G}^{-+} + i\mathcal{G}^{--}.$$

The first sign is the eigenvalue of θ and the second is the eigenvalue of θ'. So

$$\mathcal{K} = \mathcal{G}^{++} + \mathcal{G}^{+-}, \qquad \mathcal{P} = \mathcal{G}^{-+} + \mathcal{G}^{--}.$$

The Lie algebra

$$\mathcal{G}' = \mathcal{G}^{++} + i\mathcal{G}^{+-} + i\mathcal{G}^{-+} + \mathcal{G}^{--}$$

is another non compact form of $\mathcal{G}^c$ (corresponding to the Cartan involution $\theta' \circ \theta$). The maximal compact subalgebra is $\mathcal{G}^{++} + \mathcal{G}^{--}$. We take the decomposition

$$\mathcal{L}_0 = \mathcal{G}^{++} + i\mathcal{G}^{+-}, \qquad \mathcal{P}' = i\mathcal{G}^{-+} + \mathcal{G}^{--}.$$

$\mathcal{L}_0$ is a non compact real form of the complexification of $\mathcal{K}$.

In the example of so(d), we can take $\theta = \theta_{d-2,2}$ and $\theta' = \theta_{s,t}$, $s + t = d$. Then

$$\mathcal{G}' = \mathrm{so(s,t)}, \qquad \mathcal{L}_0 = \mathrm{so(s-1,t-1)} \oplus \mathrm{so}(1,1)$$

and the bifundamental representation $(\mathbf{d-2}, \mathbf{2})$ splits into two irreducible representations, whith charges ± 1 with respect to $\mathrm{so}(1,1)$, $(\mathbf{d-2})^{+1} \oplus (\mathbf{d-2})^{-1}$. This is in fact the splitting of the conformal algebra with respect to the Lorentz subalgebra times the dilatation. The vector space $\mathcal{P}'$ contains the translations P_μ and the conformal boosts K_μ.

The Lie algebra $\mathcal{G}' = \mathrm{so(s,t)}$ has a Lie algebra grading,

$$\mathcal{G}' = \mathcal{L}_{-1} + \mathcal{L}_0 + \mathcal{L}_{+1},$$

being the grade the charge with respect to $\mathrm{so}(1,1)$. $\mathcal{L}_{\pm 1}$ are abelian subalgebras and $\mathcal{L}_0 \circledS \mathcal{L}_{\pm 1}$ is the Poincaré algebra.

The same procedure can be applied to all the $\mathrm{Spin}(s,t)$ algebras in Table 1, and a grading is always found with respect to a $\mathrm{so}(1,1)$ subalgebra, which

can always be identified with the dilatation of the conformal group embedded into the Spin(s,t) algebra. $\mathcal{L}_0 \circledS \mathcal{L}_{\pm 1}$ is a semidirect sum of algebras that generalizes the super Poincaré algebra. $\mathcal{L}_{\pm 1}$ are abelian algebras containing the translations and central charges. These decompositions are listed in [23] for all the simple Lie algebras. In Table 3 we give $\mathcal{L}_0$ for all the Spin(s,t) algebras.

Spin(s,t) algebra	$\mathcal{L}_0$	Fundamental Representation
$\mathrm{so(n,n)}$	$\mathrm{sl(n,\mathbb{R})} \oplus \mathrm{so}(1,1)$	$\mathbf{2n} = (\mathbf{n})^{1/2} \oplus (\mathbf{n}')^{-1/2}$
$\mathrm{so}^*(4\mathrm{n})$	$\mathrm{su}^*(2\mathrm{n}) \oplus \mathrm{so}(1,1)$	$\mathbf{4n} = (\mathbf{2n})^{1/2} \oplus (\mathbf{2n}')^{-1/2}$
$\mathrm{so(2n,\mathbb{C})}$	$\mathrm{gl(n,\mathbb{C})}$	$\mathbf{2n} = (\mathbf{n})^{1/2} \oplus (\mathbf{n}')^{-1/2}$
$\mathrm{sp(2n,\mathbb{R})}$	$\mathrm{sl(n,\mathbb{R})} \oplus \mathrm{so}(1,1)$	$\mathbf{2n} = (\mathbf{n})^{1/2} \oplus (\mathbf{n}')^{-1/2}$
$\mathrm{usp(2n,2n)}$	$\mathrm{su}^*(2\mathrm{n}) \oplus \mathrm{so}(1,1)$	$\mathbf{4n} = (\mathbf{2n})^{1/2} \oplus (\mathbf{2n}')^{-1/2}$
$\mathrm{sp(2n,\mathbb{C})}$	$\mathrm{gl(2n,\mathbb{C})}$	$\mathbf{2n} = (\mathbf{n})^{1/2} \oplus (\mathbf{n}')^{-1/2}$
$\mathrm{sl(2n,\mathbb{R})}$	$\mathrm{sl(n,\mathbb{R})} \oplus \mathrm{sl(n,\mathbb{R})} \oplus$ so(1,1)	$\mathbf{2n} = (\mathbf{n},1)^{1/2} \oplus (1,\mathbf{n}')^{-1/2}$ $\mathbf{2n}' = (\mathbf{n}',1)^{-1/2} \oplus (1,\mathbf{n})^{1/2}$
$\mathrm{su}^*(4\mathrm{n})$	$\mathrm{su}^*(2\mathrm{n}) \oplus \mathrm{su}^*(2\mathrm{n}) \oplus$ so(1,1)	$\mathbf{4n} = (\mathbf{2n},1)^{1/2} \oplus (1,\mathbf{2n}')^{-1/2}$ $\mathbf{4n}' = (\mathbf{2n}',1)^{-1/2} \oplus (1,\mathbf{2n})^{1/2}$
$\mathrm{su(n,n)}$	$\mathrm{sl(n,\mathbb{C})} \oplus$ so(1,1)	$\mathbf{2n} = (\mathbf{n})^{1/2} \oplus (\bar{\mathbf{n}}')^{-1/2}$ $\mathbf{2\bar{n}} = (\bar{\mathbf{n}})^{1/2} \oplus (\mathbf{n}')^{-1/2}$

Table 3: so$(1,1)$-grading.

When the Spin(s,t) algebra is an orthogonal algebra, $\mathcal{L}_{\pm 1}$ are in the two-fold antisymmetric representation of $\mathcal{L}_0$. When the Spin(s,t) algebra is a symplectic algebra, then $\mathcal{L}_{\pm 1}$ are in the two-fold symmetric representation of $\mathcal{L}_0$. When the Spin(s,t) algebra is a linear algebra, then $\mathcal{L}_{\pm 1}$ are in the bifundamental representation of $\mathcal{L}_0$.

The spinor representation of so(s,t), $S_{(s,t)}$ decomposes as

$$S_{(s,t)} \xrightarrow[\mathrm{so}(s-1,t-1)\oplus\mathrm{so}(1,1)]{} S^{1/2}_{(s-1,t-1)} \oplus S^{-1/2}_{(s-1,t-1)}. \tag{3}$$

If D is even, then a chiral representation decomposes into two representations with opposite chirality.

$\mathrm{so(s-1,t-1)}$ is embedded into $\mathcal{L}_0$. When promoting the spinor representation of so(s, t) to the fundamental representation of the Spin(s,t) algebra,

the splitting (3) corresponds to the splitting under $\mathcal{L}_0$, which is given also in Table 3. $\mathbf{n}'$ denotes the dual representation of $\mathbf{n}$.

In fact, one can check that $\mathcal{L}_0$ contains in each case, the full Spin($s-1, t-1$)-algebra. The embeddings are given in Table 4.

Real case	Quaternionic case	Complex case
$\mathrm{sl(2n,\mathbb{R}) \supset so(n,n)}$	$\mathrm{su^*(2n) \supset so^*(2n)}$	$\mathrm{sl(2n,\mathbb{C}) \supset so(2n,\mathbb{C})}$
$\mathrm{sl(2n,\mathbb{R}) \supset sp(2n,\mathbb{R})}$	$\mathrm{su^*(2n) \supset usp(n,n)}$	$\mathrm{sl(2n,\mathbb{C}) \supset sp(2n,\mathbb{C})}$

Table 4: Embedding of Spin(s, t) algebras.

The superalgebras of Table 2 have also a so(1, 1) grading

$$\mathcal{S} = \mathcal{L}_{-1} + \mathcal{S}_{-1/2} + \mathcal{L}_0 + \mathcal{S}_{+1/2} + \mathcal{L}_{+1}.$$

In the case of extended supersymmetry, $\mathcal{L}_0$ contains also the R-symmetry factor. The spaces $\mathcal{L}^{\pm 1}$ have a simple meaning in terms of the γ-matrices of $\mathrm{so(s-1, t-1)}$. From the grading properties and the simplicity of the superalgebra, it is clear that

$$\{\mathcal{S}_{\pm 1/2}, \mathcal{S}_{\pm 1/2}\} = \mathcal{L}_{\pm 1}.$$

In fact, $\mathcal{L}_{\pm}$ are irreducible representations of $\mathcal{L}_0$. The anticommutator

$$\{Q_{\pm 1/2}, Q_{\pm 1/2}\}$$

can in general be written as in (1), and the term γ^μ ($\mu = 1, \ldots, s+t-2$) appears since it corresponds to the momentum (with grade 1). It follows that in the r.h.s of (1) will appear only terms corresponding to matrices $\gamma^{[\mu_1 \cdots \mu_m]}$ with the same symmetry properties as γ^μ. In fact, from dimensional considerations, all of these terms appear. For $s+t = 7, 8, 9$ mod 8, the γ^μ are antisymmetric and $\dim(\mathcal{L}_\pm) = \frac{n(n-1)}{2}$. For $s+t = 3, 4, 5$ mod 8, they are symmetric and $\dim(\mathcal{L}_\pm) = \frac{n(n+1)}{2}$. For $s+t = 2, 6$ mod 8, the relevant anticommutator is a left spinor with a right spinor. In this case the morphisms (coefficients in the r.h.s. of (1)) have no definite symmetry and $\dim(\mathcal{L}_\pm) = n^2$.

4 The orthosymplectic algebra and the maximal central extension of Poincaré supersymmetry

Let $2n$ be the real dimension of a spinor representation and N the number of such spinors present in the superalgebra. The spinor charges are denoted

$$Q^i_\alpha, \qquad i = 1, \dots N, \quad \alpha = 1, \dots 2n.$$

Since the anticommutator $\{Q^i_\alpha, Q^j_\beta\}$ is symmetric, the biggest simple superalgebra containing only these odd generators (maximal Spin(s,t) algebra in the language of Ref. [13]) is osp$(1|2\mathrm{nN}, \mathbb{R})$, with bosonic part $\hat{\mathcal{L}} = \mathrm{sp}(2\mathrm{nN}, \mathbb{R})$. It is clear that $\mathcal{L}$ contains as a subalgebra the Spin(s,t) algebra plus the R-symmetry,

$$\mathcal{L} \oplus \text{R–symmetry } \subset \hat{\mathcal{L}}.$$

We want to show that the so(1,1) grading of $\mathcal{L}$ extends to $\hat{\mathcal{L}}$ and to osp(1|2nN). We have that

$$\hat{\mathcal{L}} = \hat{\mathcal{L}}_{+1} + \hat{\mathcal{L}}_0 + \hat{\mathcal{L}}_{-1},$$

where $\hat{\mathcal{L}}_0 = \mathrm{sl}(\mathrm{nN}, \mathbb{R}) \oplus \mathrm{so}(1,1)$ and $\hat{\mathcal{L}}_{\pm 1}$ are in the two-fold symmetric representation of $\mathrm{sl}(\mathrm{nN}, \mathbb{R})$ with charges ± 1 with respect to $\mathrm{so}(1,1)$. To show that this grading is compatible with the one of the Spin(s,t) algebra, we have to show that

$$\mathcal{L}_0 \oplus \text{R–symmetry} \subset \hat{\mathcal{L}}_0.$$

We consider the complex linear algebra

$$\mathrm{sl}(\mathrm{nN}, \mathbb{C}) \simeq \mathrm{gl}(\mathrm{n}, \mathbb{C}) \otimes \mathrm{gl}(\mathrm{N}, \mathbb{C})/\mathbb{C}^*.$$

(The bracket in the tensor product of algebras is defined as

$$[a \otimes a', b \otimes b'] = [a, b] \otimes [a', b']).$$

One has that

$$\mathrm{sl}(\mathrm{n}, \mathbb{C}) \simeq \mathrm{sl}(\mathrm{n}, \mathbb{C}) \otimes \mathrm{Id} \subset \mathrm{sl}(\mathrm{nN}, \mathbb{C})$$
$$\mathrm{sl}(\mathrm{N}, \mathbb{C}) \simeq \mathrm{Id} \otimes \mathrm{sl}(\mathrm{N}, \mathbb{C}) \subset \mathrm{sl}(\mathrm{nN}, \mathbb{C}),$$

and that

$$\mathrm{sl}(\mathrm{n},\mathbb{C}) \oplus \mathrm{sl}(\mathrm{N},\mathbb{C}) \subset \mathrm{sl}(\mathrm{nN},\mathbb{C}).$$

Notice that the fundamental representation of $\mathrm{sl}(\mathrm{nN},\mathbb{C})$ becomes the bifundamental $(\mathbb{C}^n \otimes \mathbb{C}^N)$ of $\mathrm{sl}(\mathrm{n},\mathbb{C}) \oplus \mathrm{sl}(\mathrm{N},\mathbb{C})$. The adjoint of $\mathrm{sl}(\mathrm{nN},\mathbb{C})$ decomposes under $\mathrm{sl}(\mathrm{n},\mathbb{C}) \oplus \mathrm{sl}(\mathrm{N},\mathbb{C})$ in the adjoint of $\mathrm{sl}(\mathrm{n},\mathbb{C}) \oplus \mathrm{sl}(\mathrm{N},\mathbb{C})$ plus the tensor products of the adjoints of $\mathrm{sl}(\mathrm{n},\mathbb{C})$ and $\mathrm{sl}(\mathrm{N},\mathbb{C})$.

Let $\mathcal{G}_n \subset \mathrm{sl}(\mathrm{n},\mathbb{C})$ and $\mathcal{G}_N \subset \mathrm{sl}(\mathrm{N},\mathbb{C})$. Then we have

$$\mathcal{G}_n \oplus \mathcal{G}_N \subset \mathrm{sl}(\mathrm{nN},\mathbb{C}).$$

In particular, $\mathcal{G}_n$ and $\mathcal{G}_N$ can be orthogonal, symplectic, or the linear algebras themselves. We have just to check that the appropriate real forms are contained in $\mathrm{sl}(\mathrm{nN},\mathbb{R})$. We recall that the real form $\mathrm{sl}(\mathrm{nN},\mathbb{R})$ is obtained from a conjugation in the fundamental representation space. Moreover, $\mathrm{sl}(\mathrm{nN},\mathbb{R})$ is the set of all traceless matrices in $\mathrm{sl}(\mathrm{nN},\mathbb{C})$, commuting with a conjugation in $\mathbb{C}^{nN}$ (which can be brought by an isomorphism to the standard conjugation on $\mathbb{C}^{nN}$).

The direct sum of real forms $\mathcal{G}^r_n \oplus \mathcal{G}^r_N$ is contained in $\mathrm{sl}(\mathrm{nN},\mathbb{R})$ if the bifundamental representation of $\mathcal{G}^r_n \oplus \mathcal{G}^r_N$ commutes with a conjugation. This happens when both, $\mathcal{G}^r_n$ and $\mathcal{G}^r_N$, commute with a conjugation in their respective spaces, $\mathbb{C}^n$ and $\mathbb{C}^N$, and when they both commute with a pseudoconjugation.

If the algebras are complex, then one can see $(\mathcal{G}_{n'})_\mathbb{R}$ and $(\mathcal{G}_{N'})_\mathbb{R}$ inside a real group

$$\begin{aligned}(\mathcal{G}_{n'})_\mathbb{R} \subset \mathrm{sl}(\mathrm{n}',\mathbb{C})_\mathbb{R} \subset \mathrm{sl}(2\mathrm{n}',\mathbb{R})) \subset \mathrm{sp}(4\mathrm{n}',\mathbb{R}),\\ (\mathcal{G}_{N'})_\mathbb{R} \subset \mathrm{sl}(\mathrm{N}',\mathbb{C})_\mathbb{R} \subset \mathrm{sl}(2\mathrm{N}',\mathbb{R})) \subset \mathrm{sp}(4\mathrm{N}',\mathbb{R})\end{aligned} \tag{4}$$

with $n = 2n'$, $N = 2N'$.

The unitary algebras are embedded into complex algebras that one can see as real, and then embedded in a linear complex group as above (4).

We give the embeddings for all dimensions and signatures in Table 5. The real dimension of the spinor representation. is given in terms of D and depends on the reality properties of the spinor. For clarity, it is given in Table 6.

The grading extends to the orthosymplectic superalgebra $\hat{\mathcal{S}} = \mathrm{osp}(1|2\mathrm{m},\mathbb{R})$, where m is the number appearing in the third column of Table 5,

$$\hat{\mathcal{S}} = \hat{\mathcal{L}}_{-1} + \hat{\mathcal{S}}_{-1/2} + \hat{\mathcal{L}}_0 + \hat{\mathcal{S}}_{+1/2} + \hat{\mathcal{L}}_{+1}.$$

D_0	ρ_0	$\hat{\mathcal{L}}_0$	$\mathcal{L}_0 \oplus$ R–symmetry
1,7	1,7	$\mathrm{sl}(2\mathrm{N}2^{\frac{\mathrm{D}-3}{2}},\mathbb{R})$	$\mathrm{sl}(2^{\frac{\mathrm{D}-3}{2}},\mathbb{R}) \oplus \mathrm{sp}(2\mathrm{N},\mathbb{R})$
0	0	$\mathrm{sl}(2\mathrm{N}2^{\frac{\mathrm{D}-4}{2}},\mathbb{R})$	$\mathrm{sl}(2^{\frac{\mathrm{D}-4}{2}},\mathbb{R}) \oplus \mathrm{sp}(2\mathrm{N},\mathbb{R})$
1,7	3,5	$\mathrm{sl}(2\mathrm{N}2^{\frac{\mathrm{D}-3}{2}},\mathbb{R})$	$\mathrm{su}^*(2^{\frac{\mathrm{D}-3}{2}},\mathbb{R}) \oplus \mathrm{usp}(2\mathrm{N}-2\mathrm{q},2\mathrm{q})$
3,5	1,7	$\mathrm{sl}(\mathrm{N}2^{\frac{\mathrm{D}-3}{2}},\mathbb{R})$	$\mathrm{sl}(2^{\frac{\mathrm{D}-3}{2}},\mathbb{R}) \oplus \mathrm{so}(\mathrm{N}-\mathrm{q},\mathrm{q})$
4	0	$\mathrm{sl}(\mathrm{N}2^{\frac{\mathrm{D}-4}{2}},\mathbb{R})$	$\mathrm{sl}(2^{\frac{\mathrm{D}-4}{2}},\mathbb{R}) \oplus \mathrm{so}(\mathrm{N}-\mathrm{q},\mathrm{q})$
3,5	3,5	$\mathrm{sl}(2\mathrm{N}2^{\frac{\mathrm{D}-3}{2}},\mathbb{R})$	$\mathrm{su}^*(2^{\frac{\mathrm{D}-3}{2}}) \oplus \mathrm{so}^*(2\mathrm{N})$
4	4	$\mathrm{sl}(2\mathrm{N}2^{\frac{\mathrm{D}-4}{2}},\mathbb{R})$	$\mathrm{su}^*(2^{\frac{\mathrm{D}-4}{2}}) \oplus \mathrm{so}^*(2\mathrm{N})$
0	2,6	$\mathrm{sl}(2\mathrm{N}2^{\frac{\mathrm{D}-2}{2}},\mathbb{R}) \supset \mathrm{sl}(2\mathrm{N}2^{\frac{\mathrm{D}-4}{2}},\mathbb{C})$	$\mathrm{sl}(2^{\frac{\mathrm{D}-4}{2}},\mathbb{C}) \oplus \mathrm{sp}(2\mathrm{N},\mathbb{C})$
2,6	0	$\mathrm{sl}(\mathrm{N}2^{\frac{\mathrm{D}-2}{2}},\mathbb{R})$	$\mathrm{sl}(2^{\frac{\mathrm{D}-4}{2}},\mathbb{R}) \oplus \mathrm{sl}(2^{\frac{\mathrm{D}-4}{2}},\mathbb{R}) \oplus \mathrm{sl}(\mathrm{N},\mathbb{R})$
2,6	2,6	$\mathrm{sl}(\mathrm{N}2^{\frac{\mathrm{D}-2}{2}},\mathbb{R}) \supset \mathrm{sl}(\mathrm{N}2^{\frac{\mathrm{D}-4}{2}},\mathbb{C})$	$\mathrm{sl}(\mathrm{N}2^{\frac{\mathrm{D}-4}{2}},\mathbb{C}) \oplus \mathrm{su}(\mathrm{N}-\mathrm{q},\mathrm{q})$
2,6	4	$\mathrm{sl}(2\mathrm{N}2^{\frac{\mathrm{D}-2}{2}},\mathbb{R})$	$\mathrm{su}^*(2^{\frac{\mathrm{D}-4}{2}}) \oplus \mathrm{su}^*(2^{\frac{\mathrm{D}-4}{2}}) \oplus \mathrm{su}^*(2\mathrm{N})$
4	2,6	$\mathrm{sl}(\mathrm{N}2^{\frac{\mathrm{D}-2}{2}},\mathbb{R}) \supset \mathrm{sl}(\mathrm{N}2^{\frac{\mathrm{D}-4}{2}},\mathbb{C})$	$\mathrm{sl}(2^{\frac{\mathrm{D}-4}{2}},\mathbb{C}) \oplus \mathrm{so}(\mathrm{N},\mathbb{C})$
0	4	$\mathrm{sl}(2\mathrm{N}2^{\frac{\mathrm{D}-4}{2}},\mathbb{R})$	$\mathrm{su}^*(2^{\frac{\mathrm{D}-4}{2}},\mathbb{C}) \oplus \mathrm{usp}(2\mathrm{N}-2\mathrm{q},2\mathrm{q})$

Table 5: Graded embeddings

To see this, it is enough to give the decomposition of the fundamental representation of $\mathrm{sp}(2\mathrm{m},\mathbb{R})$ with respect to $\mathrm{sl}(\mathrm{m},\mathbb{R}) \oplus \mathrm{so}(1,1)$,

$$(\mathbf{2m}) \xrightarrow[\mathrm{sl}(\mathrm{m},\mathbb{R})\oplus\mathrm{so}(1,1)]{} (\mathbf{m})^{1/2} \oplus (\mathbf{m}')^{-1/2}.$$

Finally, $\hat{\mathcal{S}}_{+1/2} + \hat{\mathcal{L}}_{+1}$ is a superalgebra which is the maximal central extension of the supertranslation algebra.

ρ_0(odd)	real dim(S)	reality	ρ_0(even)	real dim($S^\pm$)	reality
1	$2^{(D-1)/2}$	$\mathbb{R}$	0	$2^{D/2-1}$	$\mathbb{R}$
3	$2^{(D+1)/2}$	$\mathbb{H}$	2	$2^{D/2}$	$\mathbb{C}$
5	$2^{(D+1)/2}$	$\mathbb{H}$	4	$2^{D/2}$	$\mathbb{H}$
7	$2^{(D-1)/2}$	$\mathbb{R}$	6	$2^{D/2}$	$\mathbb{C}$

Table 6: Real dimensions of spinor representations

Acknowledgements

S. F. wants to thank the INFN, Sezione di Torino for its kind hospitality during the completion of this work. The work of S. F. has been supported in part by the European Commission RTN network HPRN-CT-2000-00131, (Laboratori Nazionali di Frascati, INFN) and by the D.O.E. grant DE-FG03-91ER40662, Task C.

References

[1] W. Nahm, "Supersymmetries and their representations", *Nucl. Phys.* **B135** 149 (1978).

[2] J. Strathdee, "Extended Poincare Supersymmetry", *Int. J. Mod. Phys.* **A2** (1) 273 (1987).

[3] P. Horava and E. Witten, "Heterotic and Type I String Dynamics from Eleven Dimensions", *Nucl. Phys.* **B460** 506 (1996).

[4] O. Aharony, S. S. Gubser, J. Maldacena, H. Ooguri and Y. Oz, "Large N Field Theories, String Theory and Gravity". *Phys. Rept.* **323**, 183 (2000).

[5] P.K. Townsend, "M(embrane) theory on T^9", *Nucl. Phys. Proc. Suppl.* **68** 11 (1998); "p-Brane Democracy", in the proceedings of the *March 95 PASCOS/John Hopkins conference*. hep-th/9507048.

[6] M. Günaydin, "Unitary Supermultiplets of OSp(1/32,R) and M-theory", *Nucl. Phys.* **B528** 432 (1998).

[7] S. Ferrara and M. Porrati, "AdS(5) Superalgebras with Brane Charges", *Phys. Lett.* **B458** 43 (1999).

[8] A. Van Proeyen, "Tools for supersymmetry", hep-th/9910030.

[9] I. Bars, "Two-Time Physics in Field Theory", *Phys. Rev.* **D62** 046007 (2000);"A case for 14 dimensions", *Phys. Lett.* **B403** 257-264 (1997); "S-Theory", *Phys. Rev.* **D55** 2373-2381 (1997).

[10] P. Horava, "M-theory as a Holographic Field Theory", *Phys. Rev.* **D59** 046004 (1999).

[11] R. D'Auria and P. Frè, "Geometric Supergravity in $D = 11$ and its Hidden Supergroup", *Nucl. Phys.* **B201** 101 (1982).

[12] J.W. Van Holten and A. Van Proeyen, "N=1 Supersymmetry Algebras in $D = 2$, $D = 3$, $D = 4$ mod(8)", *J. Phys.* **A 15** 3763 (1982).

[13] R. D'Auria, S. Ferrara, M. A. Lledó and V. S. Varadarajan, "Spinor Algebras". hep-th/0010124.

[14] P. West, "Hidden Superconformal Symmetry in M theory", JHEP 0008:007 (2000).

[15] C. Hull, "Duality and the signature of space-time". JHEP**9811**, 017 (1998).

[16] S. Ferrara, "Spinors, superalgebras and the signature of space-time". hep-th/0101123.

[17] B. Zumino, "Euclidean Supersymmetry and the Many-Instanton Problem". *Phys. Lett.* **B69** 369 (1977).

[18] W. Siegel, "The N=2 (4) String is Selfdual N=4 Yang-Mills". *Phys. Rev.* **D46** 3235 (1992); "Selfdual N=8 Supergravity as Closed N=2 (N=4) Strings". *Phys. Rev.* **D 47** 2504 (1993); "Supermulti-Instantons in Conformal Chiral Superspace". *Phys. Rev.* **D 52** 1042 (1995).

[19] D. G. McKeon, "Harmonic Superspace with Four-Dimensional Euclidean Space". *Can. J. Phys.* **78** 261 (2000); "The Simplest Superalgebras in Two, Three, Four and Five Dimensions". *Nucl. Phys.* **B591** 591 (2000).
D. G. McKeon and T. N. Sherry, "Extended Supersymmetry in Four-Dimensional Euclidean Space". *Annals Phys.* **285** 221 (2000).
F. T. Brandt, D. G. McKeon and T. N. Sherry, "Supersymmetry in 2 + 2 Dimensions". *Mod. Phys. Lett.* **A15** 1349 (2000).

[20] A. V. Belitsky, S. Vandoren and P. van Nieuwenhuizen, "Instantons, Euclidean Supersymmetry and Wick Rotations". *Phys. Lett.* **B477** 335 (2000).

[21] P. Deligne. "Notes on Spinors" in *Quantum Fields and Strings: A Course for Mathematicians.* American Mathematical Society. (1999).

[22] S. Helgason. *Differential Geometry, Lie Groups and Symmetric Spaces.* Academic Press, (1978).

[23] R. Gilmore. *Lie Groups, Lie Algebras and Some of their Applications.* John Wiley and sons, (1974).